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NONCLASSICAL ACOUSTICS

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NONCLASSICAL ACOUSTICS*

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ABSTRACT

A statistical approach to sound propagation is considered in situations where, due to the presence of large gradients of properties of the medium, the classical (deterministic) treatment of wave motion is inadequate. Mathematical methods for wave motions not restricted to small wavelengths (analogous to known methods of quantum mechanics) are used to formulate a wave theory of sound in nonuniform flows. Nonlinear transport equations for field probabilities are derived for the limiting case of noninteracting sound waves and it is postulated that such transport equations, appropriately generalized, may be used to predict the statistical behavior of sound in arbitrary flows.

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I. INTRODUCTION

In this work we address ourselves to the nonclassical treatment of sound as a wave phenomenon. By "nonclassical treatment" we shall mean an analysis based on mathematical methods developed for the description of quantum systems in cases where wave properties change appreciably over one wavelength. The objectives of this paper are to formulate a quantum-like stochastic theory of sound of arbitrary intensity propagating through a perfect gas in an arbitrary irrotational flow.

A classical approach to wave analysis is valid only if $|\nabla\lambda| \ll 1$, where λ = wavelength. Acoustics of inhomogeneous moving fluids concerns itself usually with the zero-wavelength approximation, $\nabla\lambda = 0$. An extension of the classical approach in the spirit of the WKB (Wentzel-Kramers-Brillouin) approximation, Kentzer (1975a), resulted in a dispersion relation valid for small but not necessarily negligible wavelength. However, situations where $|\nabla\lambda| = O(1)$ are not uncommon in high speed flows. This may be demonstrated on the basis of a plane wave propagating through a fluid which moves with velocity U . If the speed of sound is a , then the frequency is $\omega = (U \pm a)k$ and we have along the wave trajectory (here $k = 2\pi/\lambda$ = wavenumber)

$$\frac{dk}{dt} = - \frac{\partial \omega}{\partial x} = - k \frac{\partial}{\partial x}(U \pm a), \quad \frac{dx}{dt} = \frac{\partial \omega}{\partial k} = (U \pm a)$$

so that, in this case,

$$\frac{d\lambda}{dx} = \frac{\lambda}{U \pm a} \frac{\partial}{\partial x}(U \pm a).$$

Thus the gradient of wavelength is equal to the fractional change in the propagation velocity, $(U \pm a)$, occurring over one wavelength. Therefore a classical approach is valid if $\lambda \ll |(U \pm a)/\nabla(U \pm a)|$.

There are many applications, especially in the field of aerodynamic noise, where this inequality is violated. For example, when a sound wave propagates upstream a small change in $(U-a)$ may produce a large fractional change in $(U-a)$ especially when $U \sim a$ or at near-sonic conditions. Another situation demanding nonclassical treatment is one in which sound propagates across boundary layers or shear layers, for then $\partial U/\partial x$ is large. A very special situation arises in turbulent flows where, due to the "graininess" of the medium, $\partial U/\partial x$ is very large almost everywhere and may change sign several times in the distance of one wavelength. When vorticity or entropy gradients are present, one must consider a more general theory which would include the interactions of acoustic waves with vorticity and entropy waves. Such a theory, based on the full Navier-Stokes equations of fluid mechanics, was developed by Kentzer (1975b).

In this paper we limit ourselves to sound propagating through arbitrary potential (irrotational) flows of a perfect gas. The organization of the paper is as follows. First, we determine the normal modes of the linear part of the equation for the fluctuations around the instantaneous local mean flow, Sec. II. The normal modes, which form a doubly infinite set of orthogonal basis vectors, are used to form an integral representation of the solution of the nonlinear equation. The exact nonlinear equation is satisfied only locally on the average under the assumption that the phase of the fluctuations is a random function. Such a function is then modelled so as to preserve a wave-particle duality. A complex characteristic function ψ is introduced in Sec. III, and an equation in the form of a nonlinear, convective and dissipative Schrödinger equation for the characteristic functions is derived treating local mean flow conditions as constant. The Schrödinger-type equation is then recast in Sec. IV back into a hydrodynamic form by Madelung's transformation in which the new dependent variables are now the absolute value of the magnitude and the phase of the characteris-

tic functions. The square of the magnitude, $|\psi|^2$, is then interpreted as a probability density and the gradient of the phase, when multiplied by a constant with units of action density, is interpreted as a probability diffusion velocity. In Sec. V we show how ensemble averages (moments) of products of wavenumber components may be defined using the characteristic functions, and how the moments determine the wavenumber distribution functions. The distribution functions, in turn, are used to calculate local ensemble averages including the nonlinear wave interaction terms. Examples of averages of quantities of interest are given. Conclusions are summarized in Sec. VI. Finally, in the Appendix, we factor out the linearized convective wave equation for the two orthogonal eigensolutions into two linearly independent first order equations in the characteristic form.

II. EXPANSION IN TERMS OF NORMAL MODES OF THE LINEARIZED WAVE EQUATION

Consider an isentropic flow of a barotropic fluid governed by the equations

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u} + \frac{1}{\rho} \nabla p = 0$$

$$\frac{\partial p}{\partial t} + \bar{u} \cdot \nabla p + \rho a^2 \nabla \cdot \bar{u} = 0,$$

where p = pressure, $\rho = \rho(p)$ = mass density, $a^2 = \gamma p / \rho$, γ = ratio of specific heats.

It will be convenient to introduce a velocity potential such that the velocity vector \bar{u} is given by $\bar{u} = \nabla \phi$, and a pressure potential P defined by $P = \int dp / \rho(p) = a^2 / (\gamma - 1)$. Thus

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u} + \nabla P = 0$$

$$\frac{\partial P}{\partial t} + \bar{u} \cdot \nabla P + (\gamma - 1) P \nabla \cdot \bar{u} = 0,$$

or,

$$\frac{\partial \nabla \phi}{\partial t} + \nabla \phi \cdot \nabla \nabla \phi + \nabla P = 0$$

$$\frac{\partial P}{\partial t} + \nabla \phi \cdot \nabla P + (\gamma - 1) P \nabla \cdot \nabla \phi = 0.$$

We introduce now the mean (ensemble average) and fluctuation values, $\phi = \bar{\phi} + \phi$, $P = \bar{P} + P'$. The result is

$$\begin{aligned} & \left[\frac{\partial \bar{\phi}}{\partial t} + \bar{\phi} \cdot \nabla \bar{\phi} + \bar{P} \right] + \left[\frac{\partial \nabla \phi}{\partial t} + \bar{\phi} \cdot \nabla \nabla \phi + \nabla \phi \cdot \nabla \bar{\phi} + \nabla P' \right] \\ & + [\nabla \phi \cdot \nabla \nabla \phi] = 0 \end{aligned}$$

$$\begin{aligned} & \left[\frac{\partial \bar{P}}{\partial t} + \bar{\phi} \cdot \nabla \bar{P} + (\gamma - 1) \bar{P} \nabla \cdot \bar{\phi} \right] + \left[\frac{\partial P'}{\partial t} + \bar{\phi} \cdot \nabla P' + \nabla \phi \cdot \nabla \bar{P} \right. \\ & \left. + (\gamma - 1) (\bar{P} \nabla \cdot \nabla \phi + P' \nabla \cdot \nabla \bar{\phi}) + [\nabla \phi \cdot \nabla P' + (\gamma - 1) P' \nabla \cdot \nabla \phi] = 0. \right. \end{aligned}$$

By definition, the averages of the fluctuations vanish. We may, therefore, average the above equations to obtain differential equations for the mean motion:

$$\frac{\partial \bar{\phi}}{\partial t} + \bar{\phi} \cdot \nabla \bar{\phi} + \bar{p} + \overline{(\bar{U} \cdot \nabla \bar{\phi})} = 0 \quad (1)$$

$$\frac{\partial \bar{p}}{\partial t} + \bar{\phi} \cdot \nabla \bar{p} + (\gamma-1) \bar{p} \nabla \cdot \bar{\phi} + \overline{(\bar{\phi} \cdot \nabla \bar{p})} + (\gamma-1) \overline{(\bar{p}' \nabla \cdot \bar{\phi})} = 0 \quad (2)$$

Subtracting Eqs. (1) and (2) we obtain the differential equations for the fluctuations:

$$\begin{aligned} \frac{\partial \phi'}{\partial t} + \bar{U} \cdot \nabla \phi' + \bar{p}' &= [-\bar{\phi} \cdot \nabla \phi'] + [\overline{(\bar{\phi} \cdot \nabla \phi')}] - (\bar{\phi} \cdot \nabla \phi') \\ &= \bar{I}_1 + \bar{B}_1 \end{aligned} \quad (3)$$

$$\begin{aligned} \frac{\partial p'}{\partial t} + \bar{U} \cdot \nabla p' + (\gamma-1) \bar{p} \nabla \cdot \phi' &= [-\bar{\phi} \cdot \nabla \bar{p} - (\gamma-1) \bar{p}' \nabla \cdot \bar{U}] + [\overline{(\bar{\phi} \cdot \nabla \bar{p})}] \\ &\quad - \bar{\phi} \cdot \nabla p' + (\gamma-1) \{ \overline{(\bar{p}' \nabla \cdot \bar{\phi})} - \bar{p}' \nabla \cdot \bar{\phi} \} \\ &= \bar{I}_2 + \bar{B}_2 \end{aligned} \quad (4)$$

where $\bar{U} = \nabla \bar{\phi}$ = mean velocity vector. In Eqs. (1) and (2) the averages of the products of the fluctuations are in the nature of Reynolds stresses. If the energy of the acoustic fluctuations is small, one may neglect the quadratic terms in (1) and (2) to facilitate computations, for then Eqs. (1) and (2) uncouple from Eqs. (3) and (4), and the mean flow becomes independent of the fluctuations. In Eqs. (3) and (4) we have separated by square brackets terms linear in the fluctuations, \bar{I}_α , and terms quadratic in the fluctuations, \bar{B}_α . These terms will be referred to as the wave interaction terms. We note here that \bar{I}_α vanish if the mean flow is uniform, and \bar{B}_α vanish if the mean flow is uniform and the squares of the fluctuations are negligible, $\alpha = 1, 2$.

We digress now and consider only the linear parts of Eqs. (3) and (4), that is, we set $\bar{I}_\alpha = \bar{B}_\alpha = 0$, $\bar{U} = \text{constant}$, $\bar{p} = \text{constant}$.

In the linear case we may take as the basic solution

$$\phi = \phi_0 e^{i(\vec{x} \cdot \vec{k} - \omega t)}, \quad p' = p_0 e^{i(\vec{x} \cdot \vec{k} - \omega t)},$$

substitute in Eqs. (3) and (4), take a scalar product of Eq. (3) with \vec{U} and solve for the amplitudes ϕ_0 and p_0 :

$$\begin{vmatrix} U_j(\omega k_j - U_k k_j k_j), & -ik_j U_j \\ -(\gamma-1)\bar{p}k_k k_k, & -i(\omega - U_k k_k) \end{vmatrix} \begin{Bmatrix} \phi_0 \\ p_0 \end{Bmatrix} = 0. \quad (5)$$

Nontrivial solutions of Eq. (5) exist if and only if the determinant of the coefficient matrix vanishes. Setting the determinant equal to zero gives the familiar dispersion relations for sound, either in the quadratic or linear form:

$$(\omega - \vec{U} \cdot \vec{k})^2 = (\gamma-1)\bar{p}k^2 = a^2 k^2 \quad (6)$$

$$\text{or,} \quad \omega = \vec{U} \cdot \vec{k} \pm ak = \vec{U} \cdot \vec{k} + c_\alpha ak, \quad c_1 = 1, \quad c_2 = -1. \quad (7)$$

In the above, \vec{k} = wavenumber vector, $k = |\vec{k}|$. To the two distinct roots there correspond two eigenvectors,

$$R_{1j} = \{1, -1\}, \quad R_{2j} = \{1, 1\}, \quad R_{\alpha j} R_{\beta j} = 2\delta_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}.$$

The amplitudes ϕ_0 and p_0 for the two modes $\alpha = 1, 2$ are then given by

$$\phi_0 = \frac{A_\alpha}{ak} R_{\alpha 1}, \quad p_0 = -iA_\alpha R_{\alpha 2}.$$

Superposing the solutions and assuming a continuous spectrum, we may write

$$\phi = \Sigma \int \frac{A_\alpha(\vec{k})}{ak} R_{\alpha 1} e^{is_\alpha(\vec{k})} d\vec{k}, \quad p' = -i\Sigma \int A_\alpha(\vec{k}) R_{\alpha 2} e^{is_\alpha(\vec{k})} d\vec{k} \quad (6)$$

where $\int \dots d\vec{k} = \int_{-\infty}^{+\infty} dk_1 dk_2 dk_3$, $\Sigma = \sum_{\alpha=1}^{\alpha=2}$, $s_\alpha = \vec{x} \cdot \vec{k} - \omega_\alpha t$.

Returning now to the nonlinear equations for the fluctuations, Eqs. (3) and (4), we may expand the dependent variables using the doubly infinite set of orthogonal vectors $R_{\alpha j} e^{i s}$ as basis vectors and use the expressions (6) provided that the wave amplitudes A_{α} are allowed to depend on space and time, $A_{\alpha} = A_{\alpha}(\bar{x}, t, \bar{k})$. Equations (3) and (4), multiplied by various powers of $A_{\alpha}^* R_{\alpha j} e^{-i s}$ and integrated over \bar{k} , would generate an infinite set of integro-differential equations for amplitude correlations of various orders. Due to the nonlinear terms B_{α} in (3) and (4), the infinite set would be open, i.e., there would be always more unknown correlations appearing in the interaction terms than the number of equations. In order to avoid generating an open hierarchy of correlation equations we assume that the amplitudes A_{α} vary slowly in space and have a random phase. We will set

$$A_{\alpha}(\bar{x}, t, \bar{k}) \propto A_{\alpha}(\bar{k}) e^{-i \tilde{\omega}_{\alpha} t} \quad (7)$$

where $\tilde{\omega}_{\alpha}$ is a complex random function. The assumption (7) amounts to modifying the frequency ω_{α} by an addition of the random function $\tilde{\omega}_{\alpha}$. With this modification of frequency, substitution of expressions (6) into Eqs. (3) and (4) gives

$$\begin{aligned} \sum_{\alpha} \int \frac{\bar{k}}{a k} \tilde{\omega}_{\alpha} A_{\alpha} R_{\alpha 1} \exp[i(\bar{x} \cdot \bar{k} - \omega_{\alpha} t - \tilde{\omega}_{\alpha} t)] d\bar{k} &= \bar{I}_1 + \bar{B}_1 \\ - \sum_{\alpha} \int \tilde{\omega}_{\alpha} A_{\alpha} R_{\alpha 2} \exp[i(\bar{x} \cdot \bar{k} - \omega_{\alpha} t - \tilde{\omega}_{\alpha} t)] d\bar{k} &= I_2 + B_2. \end{aligned}$$

We now multiply the first of the above equations by $\frac{\bar{k}'}{a k'} A_{\beta}^* R_{\beta 1} \exp[-i(\bar{x} \cdot \bar{k}' - \omega_{\beta}' t - \tilde{\omega}_{\beta}' t)]$, and the second by $A_{\beta}^* R_{\beta 2} \exp[-i(\bar{x} \cdot \bar{k}' - \omega_{\beta}' t - \tilde{\omega}_{\beta}' t)]$, subtract, and integrate with respect to \bar{k}' . Due to the orthogonality of the eigenvectors $R_{\alpha j}$ we have

$$\begin{aligned} 2 \int \tilde{\omega}_{\beta}' A_{\beta}^2 d\bar{k} &= a \int (\bar{I}_1 + \bar{B}_1) \cdot \frac{\bar{k}'}{k'} A_{\beta}^*(\bar{k}') R_{\beta 1} \exp[-i(\bar{x} \cdot \bar{k}' - \omega_{\beta}' t - \tilde{\omega}_{\beta}' t)] d\bar{k}' \\ &- \int (I_2 + B_2) A_{\beta}^*(\bar{k}') R_{\beta 2} \exp[-i(\bar{x} \cdot \bar{k}' - \omega_{\beta}' t - \tilde{\omega}_{\beta}' t)] d\bar{k}' \quad (8) \end{aligned}$$

where A_{β}^* = complex conjugate of A_{β} , $\omega' = \omega(\vec{k}')$, $\tilde{\omega}' = \tilde{\omega}(\vec{k}')$. If the amplitudes $A_{\beta}(\vec{k})$ are known, even approximately, as functions of \vec{k} , the indicated integrals may be evaluated subject to appropriate resonance conditions.

We remark here that the approximation (7) accomplishes three things at the same time, 1st, we satisfy formally the fluid dynamical equations for the fluctuations, Eqs. (3)-(4), by considering them to be defining equations for $\tilde{\omega}_{\alpha}$, 2nd, we can account for the nonlinear wave interactions in the average at a given point and instant of time, and, 3rd, we avoid the closure problem by not using Eqs. (3)-(4) as kinematical equations for the time evolution of the amplitudes A_{α} and their correlations.

Since I_{α} and B_{α} are, respectively, linear and bilinear in the fluctuations ϕ and P' , it is evident that $\tilde{\omega}_{\alpha}$ will have separate contributions from I_{α} (two-wave interactions) and from B_{α} (three-wave interactions). Due to the products of exponentials appearing under the integral signs in Eq. (8) the wave interactions are interpreted as two- and three-wave resonances. The resonance conditions, analogous to equations of conservation of quasi-particle momentum and energy, are of the form

$$\vec{k} = \vec{k}', \quad \omega_{\alpha}(\vec{k}) + \tilde{\omega}_{\alpha}(\vec{k}) = \omega_{\beta}(\vec{k}') + \tilde{\omega}_{\beta}(\vec{k}')$$

for pairs of interacting waves, and

$$\vec{k} = \vec{k}' + \vec{k}'', \quad \omega_{\alpha}(\vec{k}) + \tilde{\omega}_{\alpha}(\vec{k}) = \omega_{\beta}(\vec{k}') + \tilde{\omega}_{\beta}(\vec{k}') + \omega_{\gamma}(\vec{k}'') + \tilde{\omega}_{\gamma}(\vec{k}'')$$

for triads of interacting waves, $\alpha, \beta, \gamma = 1, 2$.

A preliminary study of wave interaction integrals in Eq. (8) indicates that, in absence of randomness ($\tilde{\omega}_{\alpha} = 0$), only waves with parallel wavenumber vectors may interact and that an addition of even the smallest viscous terms would eliminate all interactions among acoustic waves of finite wavelength. However, the randomness of the medium ($\tilde{\omega}_{\alpha} \neq 0$) makes the resonance an effective mechanism

for the redistribution of the spectral energies.

In order to solve Eq. (8) for $\tilde{\omega}_B$ more easily we will assume the wavenumber dependence of $\tilde{\omega}_B$. The lowest order nontrivial polynomial approximation to $\tilde{\omega}_B(\vec{k})$ is a quadratic expression,

$$\tilde{\omega}_B(\vec{k}) = \langle \tilde{\omega}_B \rangle \frac{k^2}{\langle k^2 \rangle} = (\epsilon_B - i\epsilon_L)k^2, \quad (9)$$

which expression has an advantage of being analogous to a Hamiltonian quadratic in momenta, and the advantage of affording an interpretation of wave resonances as interactions among quasi-particles (wave packets).

With the expression (9) for $\tilde{\omega}_B(\vec{k})$ we reduce the integral Eq. (8) to an algebraic equation for the complex function $(\epsilon_B - i\epsilon_L)$:

$$\begin{aligned} \epsilon_B - i\epsilon_L = & \{ a \int (\vec{I}_1 + \vec{B}_1) \cdot \frac{\vec{k}'}{k} \Lambda_B^*(\vec{k}') R_{B1} \exp[-i(\vec{x} \cdot \vec{k} - \omega_B' t - \tilde{\omega}_B' t)] d\vec{k}' \\ & + \int (\vec{I}_2 + \vec{B}_2) \Lambda_B^*(\vec{k}') R_{B2} \exp[-i(\vec{x} \cdot \vec{k} - \omega_B' t - \tilde{\omega}_B' t)] d\vec{k}' \} \\ & \times [2 \int k'^2 f_B(\vec{k}') d\vec{k}']^{-1} \end{aligned}$$

where $f_B(\vec{k}) \sim |\Lambda_B|^2$ = wavenumber distribution function of the B- mode.

III. CHARACTERISTIC FUNCTIONS

$$\text{Let } \psi_{\alpha} \equiv \int A_{\alpha} \exp[i(\vec{x} \cdot \vec{k} - \omega_{\alpha} t - \tilde{\omega}_{\alpha} t)] d\vec{k} = \int A_{\alpha} e^{is_{\alpha}} d\vec{k}$$

where $s_{\alpha} = \vec{x} \cdot \vec{k} - \omega_{\alpha} t - \tilde{\omega}_{\alpha} t$ is the phase. If A_{α} is a function of \vec{k} only, then

$$\frac{\partial \psi_{\alpha}}{\partial t} = -i \int (\omega_{\alpha} + \tilde{\omega}_{\alpha}) A_{\alpha}(\vec{k}) e^{is_{\alpha}} d\vec{k}$$

$$\nabla \psi_{\alpha} = i \int \vec{k} A_{\alpha} e^{is_{\alpha}} d\vec{k}, \quad \nabla^2 \psi_{\alpha} = - \int k^2 A_{\alpha} e^{is_{\alpha}} d\vec{k}.$$

Consequently, to the dispersion relation

$$[\omega + \tilde{\omega} - (\zeta - i\epsilon)k^2 - \vec{U} \cdot \vec{k}]^2 - a^2 k^2 = 0$$

there corresponds the differential equation

$$[i(\frac{\partial}{\partial t} + \vec{U} \cdot \nabla) + (\zeta - i\epsilon)\nabla^2]^2 \psi + a^2 \nabla^2 \psi = 0,$$

which has to be satisfied by the characteristic function ψ_{α} for both modes $\alpha = 1$ and $\alpha = 2$.

The objections to the use of the wave equation for ψ_{α} are the need for two initial conditions for the characteristic function of each mode and the difficulty of separating the magnitude and phase of the characteristic function in an equation of second order. When $\tilde{\omega} = 0$, the equation reduces to the wave equation. As shown in the Appendix, if \vec{U} and a are kept constant the wave equation may be replaced by a system of two equations of first order in a characteristic form:

$$[\frac{\partial}{\partial t} + (\vec{U} + c_{\alpha} \vec{a} \vec{n}) \cdot \nabla] \psi_{\alpha} = 0, \quad c_1 = 1, \quad c_2 = -1,$$

with the arbitrary unit vector \vec{n} being the space component of the characteristic normal. This result suggests the following heuristic approach. We first factor out the dispersion relation into

two factors, $\omega_\alpha = \bar{U} \cdot \bar{k} + c_\alpha a k + (\xi_\alpha - i\zeta_\alpha) k^2$, and write $k = \bar{n} \cdot \bar{k}$ where $\bar{n} = \bar{k}/k =$ unit vector in the direction of \bar{k} . Thus the differential operator corresponding to k is $-i\bar{n} \cdot \nabla$. In anticipation of a generalization of the results to nonuniform flows, we will first symmetrize the dispersion relation. Subject to verification by experiment, we shall take

$$\omega_\alpha = \frac{1}{2}(\bar{U} \cdot \bar{k} + \bar{k} \cdot \bar{U}) + \frac{c_\alpha}{2}(a\bar{n} \cdot \bar{k} + \bar{n} \cdot \bar{k}a) - [\bar{k} \cdot (\xi_\alpha - i\zeta_\alpha) \bar{k}] = 0$$

to which there corresponds the differential equation for the characteristic function of the α -mode:

$$\begin{aligned} i \frac{\partial \psi_\alpha}{\partial t} + \frac{1}{2}[\bar{U} \cdot \nabla \psi_\alpha + \nabla \cdot (\bar{U} \psi_\alpha)] + \frac{1}{2}c_\alpha[a\bar{n} \cdot \nabla \psi_\alpha + \bar{n} \cdot \nabla(a\psi_\alpha)] \\ + \nabla \cdot [(\xi_\alpha - i\zeta_\alpha) \nabla \psi_\alpha] = 0. \end{aligned}$$

This may be rewritten in the form of a nonlinear, convective and dissipative Schrödinger equation,

$$\begin{aligned} \frac{\partial \psi_\alpha}{\partial t} + (\bar{U} + c_\alpha a \bar{n}) \cdot \nabla \psi_\alpha = \nabla \cdot (\xi_\alpha \nabla \psi_\alpha) + i \nabla \cdot (\zeta_\alpha \nabla \psi_\alpha) \\ - \frac{1}{2} \psi_\alpha \nabla \cdot [\bar{U} + c_\alpha a \bar{n}]. \end{aligned} \quad (10)$$

We observe that ξ_α plays the rôle of a diffusion coefficient while ζ_α is an analog of the Planck constant $\hbar = h/2\pi$. The units of ζ_α are (frequency) \times (length)² = (energy) \times (time)/(mass), or $\zeta_\alpha =$ specific action (action per unit mass). By analogy to \hbar , ζ_α will determine the local proportionality (scale) factor of wave and mechanical attributes in the wave-particle duality. Classical limit is obtained for ζ_α and ξ_α approaching zero. This limit corresponds to $I_\alpha = B_\alpha = 0$, the case of steady uniform mean flow with negligibly small amplitudes of the fluctuations.

IV. PROBABILITY TRANSPORT EQUATIONS

We shall put now the Schrödinger-type equation (10) for the complex characteristic function ψ_α into Madelung's hydrodynamical form, which form remains nonlinear even when \bar{U} and a are constant.

Let $\psi_\alpha = P_\alpha^{1/2} e^{iS_\alpha}$ where $P_\alpha = \psi_\alpha \psi_\alpha^* = |\psi_\alpha|^2 = \int \Lambda_\alpha \Lambda_\alpha^* d\bar{k} = \int f_\alpha(\bar{k}) d\bar{k}$ = probability density, and where $S_\alpha = \int \bar{V} \cdot d\bar{x}$, $\nabla \times \bar{V} \neq 0$ in general. Note that the units of \bar{V} are (length)⁻¹ and that $\zeta_\alpha \bar{V}_\alpha$ = velocity. Thus \bar{V} will be referred to as the "probability velocity" with ζ_α being the scale factor.

Substituting the polar form $\psi = P^{1/2} e^{iS}$ into Eq. (10), dividing by ψ_α and separating the real and imaginary parts, one obtains two nonlinear equations

$$\begin{aligned} \frac{\partial P}{\partial t} + (\bar{U} + c_\alpha a \bar{n} + 2\zeta_\alpha \bar{V}) \cdot \nabla P_\alpha &= \nabla \cdot (\xi_\alpha \nabla P_\alpha) \\ &- \{ \nabla \cdot (\bar{U} + c_\alpha a \bar{n}) + 2\zeta_\alpha \cdot (\zeta_\alpha \bar{V}) + \xi_\alpha [\frac{1}{2} (\nabla \ln P_\alpha)^2 + v_\alpha^2] \} P_\alpha \end{aligned} \quad (11)$$

$$\begin{aligned} \frac{\partial S}{\partial t} + (\bar{U} + c_\alpha a \bar{n}) \cdot \nabla S_\alpha &= \nabla \cdot [\xi_\alpha \bar{V}_\alpha + \frac{1}{2} \zeta_\alpha \nabla \ln P_\alpha] + \xi_\alpha \bar{V}_\alpha \cdot \nabla \ln P_\alpha \\ &+ \zeta_\alpha [\frac{1}{4} (\nabla \ln P_\alpha)^2 - v_\alpha^2]. \end{aligned} \quad (12)$$

Equation (11) indicates that the probability density P_α is not conserved, is convected by the fluid and diffuses relative to it with the speed of sound modified by the effects of wave interactions. The right-hand-side of (11) represents a sum of dissipation and source terms. The two modes, $\alpha = 1, 2$, couple only through the coefficients ζ_α and ξ_α .

Since the phase S of the characteristic function has no physical meaning, and since only its gradient appears in the

transport equation for P_α , we take the gradient of Eq. (12) and obtain

$$\begin{aligned} \frac{\partial \bar{V}_\alpha}{\partial t} + (\bar{U} + c_\alpha \bar{a}\bar{n}) \cdot \nabla \bar{V}_\alpha = & \nabla \{ \bar{V} \cdot (c_\alpha \bar{V}_\alpha + \frac{1}{2} c_\alpha \nabla \ln P_\alpha) + c_\alpha [\frac{1}{4} (\nabla \ln P_\alpha)^2 - \bar{V}_\alpha^2] \\ & + \frac{c_\alpha}{2} \bar{V}_\alpha \cdot \nabla \ln P_\alpha \} - \bar{V}_\alpha \cdot \nabla (\bar{U} + c_\alpha \bar{a}\bar{n}) - (\bar{U} + c_\alpha \bar{a}\bar{n}) \times (\nabla \times \bar{V}_\alpha) \\ & - \bar{V}_\alpha \times [\nabla \times (\bar{U} + c_\alpha \bar{a}\bar{n})]. \end{aligned} \quad (13)$$

We note that the hydrodynamic transport equations (11) and (13) are nonlinear even in the case of a stationary medium, $\bar{U} = 0$ and $a = \text{constant}$, and that P_α and \bar{V}_α are strongly coupled. Due to the nonlinearity of the hydrodynamical form of Eqs. (11) and (13) we feel confident to *postulate* that these equations hold even in flows with arbitrary nonconstant \bar{U} and a (in inhomogeneous media or nonuniform flows).

As a consequence of the formulation, the number of dependent variables is eight, one scalar P_α and one vector \bar{V}_α for each mode $\alpha = 1, 2$. However, the spatial distributions of P_α and \bar{V}_α will provide essential information on the spectral distribution of the fluctuations of the original two scalar unknowns.

V. STATISTICAL PROPERTIES OF FLUCTUATIONS

If we define the averages of a function of \bar{k} with respect to the distribution function $f_\alpha(\bar{k})$ as

$$\langle F(\bar{k}) \rangle_\alpha = \int F(\bar{k}) f_\alpha(\bar{k}) d\bar{k} / \int f_\alpha(\bar{k}) d\bar{k},$$

then, by repeated differentiation we may obtain averages of a product of the components of \bar{k} :

$$\langle k_{n_1}^{m_1} \dots k_{n_N}^{m_N} \rangle_\alpha = \frac{1}{2i} \left\{ \frac{1}{\psi_\alpha} \frac{\partial^K \psi_\alpha}{\partial x_{n_1}^{m_1} \dots \partial x_{n_N}^{m_N}} - \frac{1}{\psi_\alpha^*} \frac{\partial^K \psi_\alpha^*}{\partial x_{n_1}^{m_1} \dots \partial x_{n_N}^{m_N}} \right\},$$

where $K = m_1 + \dots + m_N$ = the order of the product, and where $n_1 \dots n_N$

indicates the order in which the space derivatives are taken.

The averages may be written entirely in terms of P_α and \bar{V}_α , e.g.,

$$\langle k_i \rangle = v_i, \quad \langle k_i k_j \rangle = \frac{\partial v_i}{\partial x_j} + \frac{1}{2} [v_i \frac{\partial \ln P}{\partial x_j} + v_j \frac{\partial \ln P}{\partial x_i}] \neq \langle k_j k_i \rangle.$$

If the distribution function f is also a function of \bar{x} and t , the above definition of averages may be considered to hold locally with \bar{x} and t as parameters. Since the characteristic function determines the averages of products of \bar{k} of all orders, and the moments of the distribution determine the distribution, the theory is closed. That is, if $\psi(\bar{x}_0, t_0)$ and its derivatives evaluated at \bar{x}_0 and t_0 are given, we can determine $f(\bar{x}_0, t_0, \bar{k})$ and evaluate the interaction integrals in Eq. (8). With ζ_α and ξ_α known, one may use Eqs. (11) and (13) to advance the solution to the time $t_0 + dt$.

A finite number of moments may be used to determine a finite number of terms of an expansion of the distribution function $f(\bar{k})$. For that purpose the most convenient is the Gram-Charlier expansion using Grad polynomials (multidimensional Hermite polynomials).

Such polynomials are introduced by H. Grad (1949a) and used in connection with multidimensional Gram-Charlier expansion to develop his 13-moment approximation to the kinetic theory of gases (Grad, 1949b). The convenience of the Gram-Charlier expansion lies in the fact that the coefficients of the series are the averages of the Hermite polynomials which may be written in terms of symmetrized averages of the products of \bar{k} . Thus a truncated Gram-Charlier expansion will produce a distribution function having a finite number of prescribed moments. We may write

$$f(\bar{k}) = g f d\bar{k} [1 + (H_{100} + H_{010} + H_{001}) + (H_{200} + H_{020} + H_{002} + H_{110} + H_{101} + H_{011}) + (H_{300} + H_{030} + H_{003} + H_{210} + H_{201} + H_{120} + H_{102} + H_{021} + H_{012} + H_{111}) + \dots],$$

where

$$g = \left(\frac{\sigma}{\pi}\right)^{3/2} e^{-\sigma(k_1^2 + k_2^2 + k_3^2)}, \quad \sigma = \frac{3}{2}[\langle k_1^2 \rangle + \langle k_2^2 \rangle + \langle k_3^2 \rangle]^{-1},$$

$$H_{ijk} = \frac{\bar{H}_{ijk} H_{ijk}}{i!j!k! (2\sigma)^{i+j+k}},$$

$$H_{ijk}(\bar{k}) = (-1)^{i+j+k} \frac{1}{g} \frac{\partial^{i+j+k}}{\partial k_1^i \partial k_2^j \partial k_3^k} g = \text{Hermite polynomial of order } i+j+k,$$

$$\bar{H}_{ijk} = \langle H_{ijk} \rangle = \text{average of Hermite polynomial (symmetrized)},$$

for example, $H_{200} = (2\sigma)^2 [k_1^2 - \frac{1}{2\sigma}]$, $\bar{H}_{200} = (2\sigma)^2 [\langle k_1^2 \rangle - \frac{1}{2\sigma}]$,

$$H_{110} = (2\sigma)^2 k_1 k_2, \quad \bar{H}_{110} = (2\sigma)^2 [\langle k_1 k_2 \rangle + \langle k_2 k_1 \rangle]/2.$$

Treating the Fourier components $\phi_\alpha = (A_\alpha / \alpha k) R_{\alpha 1} e^{is_\alpha}$ and $P'_\alpha = A_\alpha R_{\alpha 2} e^{is_\alpha}$ as the quantities to be averaged, and writing for

the induced velocity $\bar{u}'_{\alpha} = \nabla \phi_{\alpha} = i\bar{k}\phi_{\alpha}$, we have

$$\langle \phi \rangle = \langle \bar{u} \rangle = \langle p' \rangle = 0,$$

$$\langle \phi^2 \rangle = \langle \phi \phi^* \rangle = \frac{1}{a^2} \frac{\sum f k^{-2} A_{\alpha} A_{\alpha}^* f(\bar{k}) d\bar{k}}{\sum f f_{\alpha}(\bar{k}) d\bar{k}},$$

$$\langle u'^2 \rangle = \langle \bar{u}' \cdot \bar{u}'^* \rangle = \frac{1}{a^2} \frac{\sum f \bar{k} \cdot \bar{k} A_{\alpha} A_{\alpha}^* f(\bar{k}) d\bar{k}}{\sum f f_{\alpha}(\bar{k}) d\bar{k}} = \frac{1}{a^2} \frac{\sum f f_{\alpha}^2(\bar{k}) d\bar{k}}{\sum f f_{\alpha}(\bar{k}) d\bar{k}},$$

$$\langle p'^2 \rangle = \langle p' p'^* \rangle = \frac{\sum f A_{\alpha} A_{\alpha}^* f_{\alpha}(\bar{k}) d\bar{k}}{\sum f f_{\alpha}(\bar{k}) d\bar{k}} = \frac{\sum f f_{\alpha}^2(\bar{k}) d\bar{k}}{\sum f f_{\alpha}(\bar{k}) d\bar{k}} = a^2 \langle u'^2 \rangle.$$

If the pressure perturbations are small compared to the mean pressure, $|p'| \ll \bar{p}$, we may use the Binomial Theorem to obtain

$p' = \frac{\gamma \bar{p}}{\gamma - 1} P' / \bar{P}$, so that with $\bar{P} = \frac{\gamma}{\gamma - 1} \frac{\bar{p}}{\bar{\rho}}$ we have $p' = \bar{\rho} P'$. Then

$$\langle p'^2 \rangle = \langle p' p'^* \rangle = \bar{\rho}^2 \langle P'^2 \rangle = (\bar{\rho} a)^2 \langle u'^2 \rangle.$$

For Gaussian distribution, $f(\bar{k}) = (\frac{\sigma}{\pi})^{3/2} e^{-\sigma \bar{k}^2} \int f(\bar{k}) d\bar{k}$, we have

$$\langle p'^2 \rangle = (\frac{\sigma}{2\pi})^{3/2} \frac{p_1^2 + p_2^2}{p_1 + p_2} = \frac{1}{2} \langle p'^2 \rangle = a^2 \langle u'^2 \rangle.$$

Quantities analogous to a square of the wave amplitude are

$$\begin{aligned} |\bar{\phi}|^2 &= \bar{\phi} \bar{\phi}^* = \sum f \frac{A_{\alpha}}{a k} R_{\alpha 1} e^{i s_{\alpha} d \bar{k}} \times \sum f \frac{A_{\beta}^*}{a k} R_{\beta 1} e^{-i s_{\beta} d \bar{k}}, \\ &= \sum f \frac{|A_{\alpha}|^2}{a^2 k^2} d\bar{k} = \frac{1}{2} [\langle \bar{k}^2 \rangle_1 p_1 + \langle \bar{k}^2 \rangle_2 p_2], \end{aligned}$$

$$|\bar{P}'|^2 = \bar{P}' \bar{P}'^* = p_1 + p_2, \quad |\bar{u}'|^2 = \frac{1}{2} (p_1 + p_2), \quad |p'|^2 = \bar{\rho}^2 (p_1 + p_2).$$

A word is in order on the subject of units. The basic solutions $\phi_{\alpha} = (A_{\alpha}/ak) R_{\alpha 1} e^{i s_{\alpha}}$ and $p'_{\alpha} = A_{\alpha} R_{\alpha 2} e^{i s_{\alpha}}$ have units of

L^2T^{-1} and L^2T^{-2} , respectively, the same as the original variables, the velocity potential Φ and the pressure potential P . Thus the units of the amplitude A_α are L^2T^{-2} . Superposition of the basic solutions, their integrals over the wavenumber space, Eqs. (6), results in $\bar{\Phi}$ and \bar{P} becoming, respectively, the velocity potential density and pressure potential density, with units of $L^{-1}T^{-1}$ and $L^{-1}T^{-2}$. Since $f = |\Lambda|^2$, the units of the wavenumber distribution function $f(\bar{k})$ are L^4T^{-4} , and those of the probability density P_α are LT^{-4} . However, the expectation values of any quantity will have units the same as those of the quantity, thus the units of $\langle u'^2 \rangle$ are those of velocity square, and those of $\langle p'^2 \rangle$ are pressure square. We may add that the velocity potential Φ and the basic solution $\phi_\alpha = (A_\alpha/ak)R_{\alpha 1}e^{is_\alpha}$ have units of action density as do the coefficients ϵ_α and $\bar{\epsilon}_\alpha$.

The root-mean-square amplitudes, due to a double integration over the wavenumber space under the square root sign, should be interpreted as densities of wave properties, e.g., $\sqrt{p'p'^*}$ = wave pressure density (units of pressure/volume), $\sqrt{u'u'^*}$ = induced velocity density, etc.

VI. CONCLUSIONS

We have found that there exist situations in the acoustics of nonuniform media where classical treatment becomes inadequate. To treat such situations quantum-like statistical methods were used to derive transport equations for the probability densities which determine wavenumber distribution functions. The distribution functions permit one to evaluate ensemble averages as well as spectral distributions of physical quantities of interest.

The present theory, it may be argued (see, e.g., pp. 52-55 of Kentzer, 1975b), parallels that of quantum mechanics and obeys the *UNCERTAINTY PRINCIPLE* (existence of indeterminacy of phase or non-conservation of the number of waves), *COMPLEMENTARITY PRINCIPLE* (existence of pairs of conjugate variables, such as \bar{x} and \bar{k} , ω and t , each of which may be better defined only at the expense of a corresponding loss in the degree of definition of the other), and the *CORRESPONDENCE PRINCIPLE* (results of the theory reduce to the classical acoustics of homogeneous media).

The nonlinear effects manifest themselves in the present theory in two ways. First, the hydrodynamic form of the transport equations contains nonlinear source terms and the nonuniformity of the medium results in the coefficients of the transport equations being functions of the dependent variables. Second, the bilinear terms introduce nonlinearity at the microscopic level through the wave interaction integrals. Thus each wave of finite amplitude affects amplitudes of all other waves.

The fact that there exist two orthogonal (independent) wave solutions corresponding to a given wavenumber vector requires that a general solution or its integral representation be a sum of two independent solutions (two modes of propagation), and that two

different sets of boundary conditions may be used for each mode. Thus it is possible to impose a radiation condition at infinity, and the coexistence of the two modes at a given point may be viewed as a radiation condition at the microscopic level.

APPENDIX

Pauli's Decomposition of the Convective Wave Equation

Consider the potential equation,

$$[\frac{\partial^2 \phi}{\partial t^2} + 2\bar{u} \cdot \nabla \frac{\partial \phi}{\partial t} + \bar{u} \cdot (\bar{u} \cdot \nabla) \nabla \phi] - a^2 \nabla^2 \phi = 0.$$

This is equivalent to

$$[(\frac{\partial}{\partial t} + \bar{u}_0 \cdot \nabla)(\frac{\partial}{\partial t} + \bar{u}_0 \cdot \nabla) - (a_0 \cdot \nabla) \cdot (a_0 \cdot \nabla)]\phi = [(\frac{D_0}{D_0 \cdot t})^2 - (a_0 \cdot \nabla_0)^2]\phi = 0 \quad (A.1)$$

where the subscript ()₀ denotes that \bar{u}_0 is treated as a constant when operated upon by $\frac{D_0}{D_0 \cdot t} = \frac{\partial}{\partial t} + \bar{u}_0 \cdot \nabla$, and a_0 is kept constant when operated by ∇_0 . Equation (A.1) is the wave equation in the Euler's form for a fluid particle with an instantaneous velocity \bar{u} .

In order to factor out the second order wave operator of Eq. (A.1) into two linear operators, we write

$$(\hat{E}^2 - \hat{H}^2)\phi = 0, \quad (A.2)$$

and require that

$$\hat{O}_1 \phi = 0 \quad (A.3)$$

where \hat{E} , \hat{H} , and \hat{O}_1 are linear differential operators. Then we need an operator \hat{O}_2 such that

$$\hat{O}_2 \hat{O}_1 = (\hat{E}^2 - \hat{H}^2) = 0.$$

The operators \hat{O}_1 and \hat{O}_2 must be real and must commute. If we choose

$$\hat{O}_1 = \hat{E} + \hat{H}, \quad \text{and} \quad \hat{O}_2 = \hat{E} - \hat{H},$$

than \hat{E} and \hat{H} must also commute.

Following Pauli (see any book on the quantum theory), we represent \hat{E} and \hat{H} by matrices and ϕ by a vector. Taking

$$\hat{E} = \hat{\alpha} \frac{D_0}{4D_0 \cdot t}, \quad \hat{H} = a_0 [\hat{\alpha}_1 \frac{\partial}{\partial x} + \hat{\alpha}_2 \frac{\partial}{\partial y} + \hat{\alpha}_3 \frac{\partial}{\partial z}],$$

where $\hat{\alpha}_1, \dots, \hat{\alpha}_4$ are square matrices to be determined. We note that the operators \hat{E} and \hat{H} commute if \bar{u}_0 and a_0 are treated as constants and if $\hat{\alpha}_4$ commutes with $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\alpha}_3$. Squaring \hat{H} and requiring that $\hat{H}^2 = a_0^2 \nabla^2$, we have the following conditions

$$\hat{\alpha}_1^2 = \hat{\alpha}_2^2 = \hat{\alpha}_3^2 = \hat{I} \text{ (a unit matrix),}$$

$$\hat{\alpha}_1 \hat{\alpha}_1 = -\hat{\alpha}_1 \hat{\alpha}_1, \quad \hat{\alpha}_1 \hat{\alpha}_3 = -\hat{\alpha}_3 \hat{\alpha}_1, \quad \hat{\alpha}_2 \hat{\alpha}_3 = -\hat{\alpha}_3 \hat{\alpha}_2.$$

Thus $\hat{\alpha}_1$, $\hat{\alpha}_2$, and $\hat{\alpha}_3$ anticommute. Choosing $\hat{\alpha}_4 = \hat{I}$, we assure that $\hat{\alpha}_4$ commutes with $\hat{\alpha}_1$, $\hat{\alpha}_2$, and $\hat{\alpha}_3$, while the latter may be taken to be the Pauli spin matrices, e.g.,

$$\hat{\alpha}_1 = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \end{Bmatrix}, \quad \hat{\alpha}_2 = \begin{Bmatrix} 0 & -i \\ i & 0 \end{Bmatrix}, \quad \hat{\alpha}_3 = \begin{Bmatrix} 1 & 0 \\ 0 & -1 \end{Bmatrix}.$$

Thus the solution of Eq. (A.3) should be represented by a two-component column vector,

$$\bar{\phi} = \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix}.$$

The system (A.3) is thus equivalent to

$$\begin{aligned} \left(\frac{D_0}{D_0 t} + a_0 \frac{\partial}{\partial z} \right) \phi_1 + a_0 \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \phi_2 &= 0 \\ \left(\frac{D_0}{D_0 t} - a_0 \frac{\partial}{\partial z} \right) \phi_2 + a_0 \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \phi_1 &= 0. \end{aligned} \tag{A.4}$$

As may be readily verified, as long as \bar{u}_0 and a_0 are kept constant, ϕ_1 or ϕ_2 may be eliminated by cross differentiation so that both components of $\bar{\phi}$ satisfy the convective wave equation (A.1). With \bar{u}_0 and a_0 kept constant, Eqs. (A.4) are linear and we may consider $\bar{\phi}$ to be a sum of two independent eigensolutions,

$$\bar{\phi} = \bar{\phi}_1 + \bar{\phi}_2 = \phi_1 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + \phi_2 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \quad \bar{\phi}_1 \cdot \bar{\phi}_2 = 0,$$

and the system (A.4) decouples to yield

$$\frac{D_0 \phi_1}{D_0 t} + a_0 \frac{\partial \phi_1}{\partial z} = 0, \quad \text{and} \quad \frac{D_0 \phi_2}{D_0 t} - a_0 \frac{\partial \phi_2}{\partial z} = 0. \quad (\text{A.5})$$

By permutations of the matrices $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3$ we may single out any one of the three space coordinates x, y, z . Thus a matrix permutation is equivalent to a rotation of the coordinate axes by 90° . Since $\partial \phi_1 / \partial z = \bar{n} \cdot \nabla$ if \bar{n} is a unit vector pointing in the direction of the preferred coordinate (z -axis in this case), then an arbitrary rotation of coordinate axes generalizes (A.5) to

$$\begin{aligned} \frac{D_0 \phi_1}{D_0 t} + a_0 \bar{n} \cdot \nabla \phi_1 &= \left[\frac{\partial}{\partial t} + (\bar{u}_0 + a_0 \bar{n}) \cdot \nabla \right] \phi_1 = 0 \\ \frac{D_0 \phi_2}{D_0 t} - a_0 \bar{n} \cdot \nabla \phi_2 &= \left[\frac{\partial}{\partial t} + (\bar{u}_0 - a_0 \bar{n}) \cdot \nabla \right] \phi_2 = 0 \end{aligned} \quad (\text{A.6})$$

where \bar{n} is arbitrary. The expressions $\frac{d}{dt} = \frac{\partial}{\partial t} + (\bar{u}_0 \pm a_0 \bar{n}) \cdot \nabla$ in Eqs. (A.6) are the total derivatives along the characteristic rays with \bar{n} serving as a parameter. Thus d/dt is an Eulerian derivative written for a pseudo-particle (a wave packet) moving with a characteristic velocity (group velocity) $\bar{u}_0 \pm a_0 \bar{n}$.

Keeping \bar{n} fixed implies that the solution vector $\bar{\phi}$ coincides with a different pair of eigensolutions at each point in time and space. If we single out one such a pair initially and would like to follow their development in time, we would have to determine how the unit vector changes in time. These two choices correspond to the Heisenberg's and Schrödinger's pictures of quantum mechanics. If necessary, three equations for the components of the unit vector $\bar{n}(\bar{x}, t)$ may be obtained as follows:

$$\bar{n} \cdot \bar{n} = 1, \quad \frac{d(\bar{n} \cdot \bar{n})}{dt} = 0,$$

$$\text{and} \quad \left[\frac{\partial}{\partial t} + (\bar{u}_0 \pm a_0 \bar{n}) \cdot \nabla \right] \left[\frac{\partial}{\partial t} + (\bar{u}_0 \mp a_0 \bar{n}) \cdot \nabla \right] \phi_\alpha = \left[\left(\frac{D_0}{D_0 t} \right)^2 - (a_0 \nabla)^2 \right] \phi_\alpha = 0.$$

where $\alpha = 1, 2$. The unit vector \bar{n} may be thus determined as a function of the solution of the wave equation.

In order to derive the nonlinear convective wave equation, Eq. (A.1), all we need to postulate is that the dispersion relation for an arbitrary acoustic wave with a wavenumber vector $\vec{k} = k\vec{n}$ at a point (\vec{x}, t) , where the local velocity and speed of sound are $\vec{u}(\vec{x}, t)$ and $a(\vec{x}, t)$, is $\omega = 0$ in the inertial frame of reference which, at each point $\vec{x} = \vec{x}_0$ and time $t = t_0$, moves with different constant velocity $\vec{u}_0 \pm a_0 \vec{n}$ equal to the local group velocity of the wave. Requiring that this postulate hold for an arbitrary wave (arbitrary \vec{n}) anywhere (arbitrary \vec{x}_0) and at all times (arbitrary t_0), the dispersion relations, either $\omega - \vec{u}_0 \cdot \vec{k} \pm a_0 \vec{n} \cdot \vec{k} = 0$ or $(\omega - \vec{u}_0 \cdot \vec{k})^2 - a_0^2 k^2 = 0$, must be viewed as a result of transforming $\omega = 0$ from an inertial frame instantaneously coinciding with the wave packet and moving with the group velocity of the packet to an inertial frame common to all wave packets in all space-time. Thus $\vec{u}_0(\vec{x}, t)$ and $a_0(\vec{x}, t)$ should be treated as parametric constants when the dispersion relation is replaced by a differential operator, and should be allowed to become functions of \vec{x} and t after the differential expression is expanded. Thus, e.g., to the relation $(\omega - \vec{u}_0 \cdot \vec{k})^2 - a_0^2 k^2 = 0$ there corresponds

$$[i(\frac{\partial}{\partial t} + \vec{u}_0 \cdot \nabla)]^2 \phi + a_0^2 \nabla^2 \phi = 0.$$

Expanding, we have

$$\begin{aligned} -(\frac{\partial}{\partial t} + \vec{u}_0 \cdot \nabla)(\frac{\partial}{\partial t} + \vec{u}_0 \cdot \nabla)\phi + a_0^2 \nabla^2 \phi &= -\frac{D_0^2 \phi}{D_0 t^2} + a_0^2 \nabla^2 \phi \\ &= -[\frac{\partial^2 \phi}{\partial t^2} + 2\vec{u}_0 \cdot \frac{\partial \nabla \phi}{\partial t} + \vec{u}_0 \cdot (\vec{u}_0 \cdot \nabla) \nabla \phi] + a_0^2 \nabla^2 \phi = 0. \end{aligned}$$

Dropping the subscript ()₀, we obtain the nonlinear potential equation which, when written in terms of the operator $D_0/D_0 t$ takes the simple form of Eq. (A.1).

REFERENCES

- Grad, H. (1949a) "Note on N-Dimensional Hermite Polynomials,"
Comm. Pure Appl. Maths., Vol. 2, No. 4,
pp. 325-330.
- Grad, H. (1949b) "On the Kinetic Theory of Rarefied Gases,"
Comm. Pure Appl. Maths., Vol. 2, No. 4,
pp. 331-407.
- Kentzer, C. P. (1975a) "Amplification, Attenuation, and Dis-
persion of Sound in Inhomogeneous Flows,"
89th Meeting of A.S.A., U. of Texas at Austin,
Austin, Texas, 7-11 April 1975 (available as
School of Aeronautics & Astronautics Rep.No. 75-1,
Purdue U.).
- Kentzer, C. P. (1975b) "Wave Theory of Turbulence in Compressible
Media," School of Aeronautics & Astronautics
Rep. No. 75-2, Purdue U. (available also as
NASA CR-2671, May 1976).